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Communication

Scaling laws for transverse relaxation times

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Abstract

Simple scaling laws are useful tools in understanding the effect of changing parameters in MRI experiments. In this paper the general scaling behavior of the transverse relaxation times is discussed. We consider the dephasing of spins diffusing around a field inhomogeneity inside a voxel. The strong collision approximation is used to describe the diffusion process. The obtained scaling laws are valid over the whole dynamic range from motional narrowing to static dephasing. The dependence of the relaxation times on the external magnetic field, diffusion coefficients of the surrounding medium, and the characteristic scale of the field inhomogeneity is analyzed. For illustration the generally valid scaling laws are applied to the special case of a capillary, usually used as a model of the myocardial BOLD effect.

Keywords: Transverse relaxation; Susceptibility effects; Diffusion; Scaling

1. Introduction

The transverse relaxation times T_2 and T_2^* are fundamental quantities in MRI, especially for characterizing tissues and their properties. The influence of static magnetic field inhomogeneities on relaxation times is of special interest in understanding relaxation processes inside the voxel. It is well known that susceptibility contrasts, external magnetic fields and diffusion influences the relaxation times T_2 and T_2^*. Obviously it is useful to understand the scaling of relaxation times with respect to their parameters in order to describe effects of parameter changing in a simple way. For example scaling laws can be used to predict the effects of changing external magnetic field strength or concentration of contrast agents. It is important to know how relaxation times vary subjected to changes in these characteristic quantities. Despite this fact papers dealing with this issue are sparse in the literature. Weisskoff et al. [1] discussed the scaling behavior of the relaxation rate R_2 in the context of microscopic susceptibility variations. Starting from the Bloch-Torrey-Equation [2] they obtained two special scaling laws and verified them by extensive numerical simulation. Employing the strong collision approximation [3–5] we give a rigorous derivation of a generalized scaling law for both transverse relaxation times T_2 and T_2^*. After considering the basic model of a field inhomogeneity inside a voxel we use well known results of Bauer et al. [3] in order to obtain the scaling laws. To give an example these results are applied to a cylindrical geometry which is commonly used as a model of a vascular network.

2. Basic model

We consider an arbitrary distribution of magnetic material G inside a voxel causing a susceptibility shift $\Delta\chi = \chi_i - \chi_e$ compared to the surrounding medium with volume V (see Fig. 1). The volume fraction η of material inside the voxel is given by $\eta = G/(G+V)$. Dephasing takes place in the remaining volume V of the voxel around the magnetic perturber G, in which the diffusion of the spins is determined by the diffusion coefficient D.

The z-component of the magnetic field caused by the inhomogeneity G is given by [6,7]

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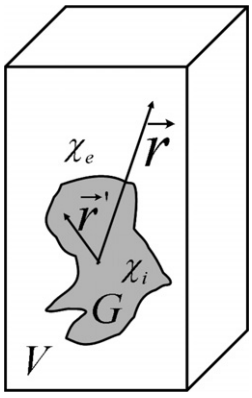

Fig. 1. Voxel containing a magnetic inhomogeneity G with susceptibility χ_i and dephasing volume V with susceptibility χ_e. All coordinates inside G are represented by primed vectors $\mathbf{r}'$ and all coordinates of V by unprimed vectors $\mathbf{r}$.

$$B_z(\mathbf{r}) = B_0 \Delta\chi \frac{\partial^2}{\partial z^2} \int_G \frac{\mathrm{d}^3\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}, \tag{1}$$

where $\mathbf{r}'$ represents the coordinates inside the volume of the perturber G and $\mathbf{r}$ the coordinates of the surrounding volume V (see Fig. 1). This means that only the field from the magnetic inhomogeneity G inside the voxel influences the dephasing of spins in V and the effects of neighboring voxels are neglected. The external magnetic field B_0 induces the local resonance frequency $\omega(\mathbf{r}) = \gamma B_z(\mathbf{r})$ which can be written as

$$\omega(\mathbf{r}) = \delta\omega f(\mathbf{r}), \tag{2}$$

where the characteristic frequency is given by

$$\delta\omega = \gamma\Delta\chi B_0, \tag{3}$$

which contains the susceptibility properties of the perturber and the external magnetic field, while the geometric function

$$f(\mathbf{r}) = \frac{\partial^2}{\partial z^2} \int_G \frac{\mathrm{d}^3\mathbf{r}\prime}{|\mathbf{r}-\mathbf{r}'|} \tag{4}$$

defines the shape of the perturber. Thus we are able to separate the susceptibility properties of the inhomogeneity G from its distribution inside the voxel. This offers the possibility to investigate the influence of each part independently.

In order to derive the scaling laws some independent approximations are necessary. First we obtain the correlation time τ by using a mean time approximation that considers a monoexponential time behavior of the correlation function. The second one is the strong collision approximation which is used to simplify the diffusion process around field inhomogeneities. The last approximation is again a mean time approximation which considers the magnetization decay as monoexponential also. As shown in previous work [8] a relation between the correlation function and the magnetization decay exists in the gaussian approximation. As it is not known if the considerations of it are fullfiled generally [9] we use the strong collision approach to describe the magnetization decay.

We start with calculating the correlation time τ which is required to obtain the magnetization decay. To investigate the dynamic property of the problem we use the two-point correlation function of the stochastic field fluctuations to which a spin is subjected. It is defined as

$$K(t) = \int_V \mathrm{d}^3\mathbf{r} \int_V \mathrm{d}^3\mathbf{r}_0\, \omega(\mathbf{r}) p(\mathbf{r}, \mathbf{r}_0, t) p(\mathbf{r}_0) \omega(\mathbf{r}_0), \tag{5}$$

where $p(\mathbf{r}, \mathbf{r}_0, t)$ is the probability density of finding a spin at point $\mathbf{r}$ after time t with the initial ($t = 0$) position $\mathbf{r}_0$, and $p(\mathbf{r}_0)$ specifies the probability density function of the equilibrium distribution. In our case the latter is identical with the spin density, which we assume to be homogeneous, i.e. $p(\mathbf{r}_0) = 1/V$. Assuming free diffusion of spins within V, the probability $p(\mathbf{r}, \mathbf{r}_0, t)$ is simply the Green's function of the diffusion equation where D is the diffusion coefficient

$$\frac{\partial}{\partial t} p(\mathbf{r}, \mathbf{r}_0, t) = D\nabla^2 p(\mathbf{r}, \mathbf{r}_0, t) \tag{6}$$

or

$$p(\mathbf{r}, \mathbf{r}_0, t) = \mathrm{e}^{tD\nabla^2} \delta(\mathbf{r} - \mathbf{r}_0), \tag{7}$$

with the reflectory boundary conditions $\partial_r p(\mathbf{r}, \mathbf{r}_0, t) = 0$ at the surface of the magnetic perturber and the voxel boundaries. In the case of permeable membranes the probability function $p(\mathbf{r}, \mathbf{r}_0, t)$ has to fulfill the radiation boundary conditions $\partial_r p(\mathbf{r}, \mathbf{r}_0, t) = k p(\mathbf{r}, \mathbf{r}_0, t)$ at the surfaces of the inhomogeneities where k is the permeability of the membrane. Insertion of the probability density Eq. (7) and $p(\mathbf{r}_0)$ into the definition of the correlation function Eq. (5), results in

$$K(t) = \frac{1}{V} \int_V \mathrm{d}^3\mathbf{r}\, \omega(\mathbf{r}) \mathrm{e}^{tD\nabla^2} \omega(\mathbf{r}). \tag{8}$$

Using Eq. (2) the correlation function at $t = 0$ is given by

$$K(0) = \frac{1}{V} \int_V \mathrm{d}^3\mathbf{r}\, \omega^2(\mathbf{r}) =: \langle \omega^2(\mathbf{r}) \rangle = \delta\omega^2 \langle f^2(\mathbf{r}) \rangle. \tag{9}$$

The result $K(0) \sim \delta\omega^2$ is the same as from Jensen and Chandra [10], Eq. (18) and in complete agreement with the more general Eq. (1) given by Sukstanskii and Yablonskiy [11]. In general the correlation function $K(t)$ does not exhibit a single exponential decay as is often assumed [12]. This hampers a simple determination of the correlation time, i.e. $K(t) \sim \mathrm{e}^{-t/\tau}$. However, a proper definition of the correlation time is to define it as the mean relaxation time of the correlation function, i.e. according to ref. [13],

$$\tau = \int_0^\infty \mathrm{d}t\, \frac{K(t)}{K(0)}. \tag{10}$$

In fact, it has been demonstrated that this definition provides the best single exponential approximation of the correlation function. A commonly used approximation for the correlation function is

$$K(t) \approx K(0) \cdot e^{-\frac{t}{\tau}}. \quad (11)$$

As shown by Jensen and Chandra [10], as well as by Sukstanskii and Yablonskiy [11], if diffusion is unrestricted, the long time limit of $K(t)$ is exactly described by an algebraic function with $K(t) \sim t^{-3/2}$. However, as we do not use any specific relation between the correlation function and the signal decay, the exact time course of the correlation function is not of particular interrest. In the strong collision approximation which is based on directly solving the Bloch-Torrey-Equation a well defined correlation time is necessary only which is given by the mean time approximation [13]. Insertion of Eqs. (8) and (9) into Eq. (10) yields

$$\tau = \frac{1}{D}\frac{1}{\langle f^2(\mathbf{r})\rangle V}\int_V d^3\mathbf{r} f(\mathbf{r})\left(-\frac{1}{\nabla^2}\right)f(\mathbf{r}), \quad (12)$$

where the expression $(1/\nabla^2)f(\mathbf{r}) = g(\mathbf{r})$ is the solution $g(\mathbf{r})$ of the inhomogeneous Laplace-Equation $\nabla^2 g(\mathbf{r}) = f(\mathbf{r})$ with the appropriate boundary conditions. Obviously this correlation time depends on the geometry function $f(\mathbf{r})$ of the perturber G and the diffusion coefficient D only. Performing the integration, the correlation time τ will depend on a characteristic size R, the diffusion coefficient D and the volume fraction η. Thus the signal properties of a certain voxel are completely described by the geometry function $f(\mathbf{r})$, the diffusion coefficient D and the characteristic frequency $\delta\omega$. Using the correlation time for a given geometry function $f(\mathbf{r})$ of the inhomogeneity G the problem can be reduced to the examination of the dependence $T_2^* = T_2^*(\tau, \delta\omega)$ as will be shown in the course of this paper.

At the end of this section it is useful to make some comments on the limitations of this model. The local magnetic field given in Eq. (1) is the first order perturbation according to the external homogeneous field B_0. Influences on the magnetization of the perturber by its own induced magnetic field are neglected. It is also assumed that the perturber consists of one material only. For this reason $\Delta\chi$ does not vary in space inside the volume G. To include systems of more than one material the perturber is considered to be composed of more than one particle and each of them is assumed to consist of just one material. Since we apply first order perturbation only the magnetic field in Eq. (1) is given by a sum of integrals, each for one single particle. This is possible as long as higher order perturbation can be neglected.

To include the effects of neighboring voxels the above method can be used. Therefore the voxel is extended to include the perturbers of these voxels but the dephasing volume remains fixed in its initial boundaries. Since calculations become extremely difficult in this case we will investigate the limits assuming a single voxel without neighbors. The single voxel approximation is valid as long as the characteristic frequency $\delta\omega$ generated by the surrounding voxels is negligible compared to the characteristic frequency on the surface of the perturber inside. For illustration we assume single sphere encircled by 12 nearest neighbors with distance $2\sqrt{2/3}s$ according to a fcc Bravais lattice. Thus we find by solving Eq. (1) for a sphere and adding up all contributions that the characteristic frequency of the perturber inside the voxel has to fulfill $\delta\omega \gg 12\delta\omega R_s^3/(2\sqrt{2/3}s)^3$ which corresponds to a volume fraction of $\eta \ll 0.5$. A similar investigation of a single capillary surrounded by six parallel capillaries with distance $2a$ gives the relation $\delta\omega \gg 6\delta\omega R_c^2/(2a)^2$, and thus $\eta \ll 0.8$. In MRI experiments with contrast agents the volume fraction of iron particles inside a macrophage used as carrier is in the order of $\eta \approx 10^{-3}$. Similar or lower concentrations are used in other experiments. Thus the single voxel approximation is satisfied.

3. General results

In order to find the relationship between the relaxation time T_2^*, the correlation time τ, as well as the characteristic frequency $\delta\omega$, the time development of the magnetization signal is studied. In the equilibrium state the magnetization is parallel to the external magnetic field, resulting in a longitudinal component only. Application of a HF-pulse induces a magnetization component $m(\mathbf{r}, t)$ in a plane transversal to the external magnetic field. The time evolution of $m(\mathbf{r}, t)$ is governed by the Bloch-Torrey-Equation [2]:

$$\frac{\partial}{\partial t}m(\mathbf{r}, t) = [D\nabla^2 + i\omega(\mathbf{r})]m(\mathbf{r}, t). \quad (13)$$

Formal time integration of the Bloch-Torrey-Equation results in

$$m(\mathbf{r}, t) = m(\mathbf{r}, 0)\exp\{[D\nabla^2 + i\omega(\mathbf{r})]t\}. \quad (14)$$

The resulting signal from the whole voxel is given by

$$M(t) = \frac{1}{V}\int_V d^3\mathbf{r} m(\mathbf{r}, 0)\exp\{[D\nabla^2 + i\omega(\mathbf{r})]t\}, \quad (15)$$

where we assume a small volume fraction of the perturber inside the voxel [14]. Hence, in the regime of static dephasing ($D = 0$) the magnetization decay is described by [14]

$$M_0(t) = \frac{1}{V}\int_V d^3\mathbf{r} m(\mathbf{r}, 0)\exp\{i\omega(\mathbf{r})t\}. \quad (16)$$

There are a few geometrically determined functions $\omega(\mathbf{r})$ for which the Bloch-Torrey-Equation can be solved analytically, e.g., when $\omega(\mathbf{r})$ is the local precession frequency in a linear gradient field. For more sophisticated functions $\omega(\mathbf{r})$, such as the field around cylinders and spheres, only numerical calculations or computer simulations may be applied to determine the time course of the transverse magnetization. Assuming that on the time scale of the local field fluctuations the magnetization $M(t)$ does not vary significantly, we may consider the local field fluctuations as stochastically independent on the time scale of the magnetization decay. This applies if the correlation time τ is much smaller than the susceptibility-induced contribution of the transverse relaxation time T_2^*. A mathematical criterion of this consideration will be given after deriving an expression of

the relaxation time T_2^*. The stochastic independence suggests replacing the diffusion operator $D\nabla^2$ in Eq. (15) by a *strong collision operator* [3].

The strong collision approximation consists of replacing the diffusion operator $D\nabla^2$ by the generator $\hat{D}$:

$$D\nabla^2 \xrightarrow[\text{collision}]{\text{strong}} \hat{D} = v(\hat{\Pi} - \hat{1}), \tag{17}$$

where $\hat{\Pi}$ denotes the projection operator onto the functional space generated by the equilibrium probability function $p(\mathbf{r})$, i.e.,

$$\hat{\Pi} g(\mathbf{r}) = p(\mathbf{r}) \int_V \mathrm{d}^3\mathbf{r} g(\mathbf{r}), \quad \hat{1} g(\mathbf{r}) = g(\mathbf{r}). \tag{18}$$

The operator $\hat{D}$ describes the stochastic field fluctuations as a stationary Markov process with a transition rate of $vp(\mathbf{r})$ between two distinct field realizations $\omega(\mathbf{r}_0) \to \omega(\mathbf{r})$, i.e., the transition rate is independent of the initial state. This means that due to the interaction with the fast field fluctuations the trajectory of the spin covers nearly the whole offresonance frequency distribution on a shorter time scale as the relaxation occurs. An approach of this type is referred to as a random phase or strong collision approximation [15]. Since the proton density in tissue is rather homogeneous, the probability density function of the equilibrium state is simply $p(\mathbf{r}) = 1/V$. It is important to note that the projection operator $\hat{\Pi} - \hat{1}$ is idempotent, i.e., $(\hat{\Pi} - \hat{1})^n = \hat{\Pi} - \hat{1}$ for $n \geqslant 1$.

If we substitute the operator $\hat{D}$ for the diffusion operator $D\nabla^2$ in the Green's function of the diffusion equation, and use the idempodency of this operator, we find for the strong collision Green's function after several rearrangement steps

$$p_{\mathrm{SC}}(\mathbf{r}, \mathbf{r}_0, t) = p(\mathbf{r}) \cdot (1 - \mathrm{e}^{-vt}) + \mathrm{e}^{-vt} \cdot \delta(\mathbf{r} - \mathbf{r}_0). \tag{19}$$

Introducing this Green's function into the expression for the correlation function, we arrive at the following equation:

$$\begin{aligned} K_{\mathrm{SC}}(t) &= \int_V \mathrm{d}^3\mathbf{r} \int_V \mathrm{d}^3\mathbf{r}_0 \omega(\mathbf{r}) p_{\mathrm{SC}}(\mathbf{r}, \mathbf{r}_0, t) p(\mathbf{r}_0) \omega(\mathbf{r}_0) \\ &= K_{\mathrm{SC}}(0) \cdot \mathrm{e}^{-vt}. \end{aligned} \tag{20}$$

The parameter v characterizes the decay rate of the magnetization and is determined self consistently in the following way [3]: the correlation time of the original stochastic process is identical to the correlation time of the strong collision approximation of this process. If we compare Eq. (20) with the correlation function $K(t) = K(0)\exp(-t/\tau)$, we find for the parameter v the relation $v = 1/\tau$. Replacing the diffusion operator in the expression for the signal from the whole sample Eq. (15) leads to

$$M(t) = \frac{1}{V} \int_V \mathrm{d}^3\mathbf{r} \mathrm{e}^{[\hat{D} + i\omega(\mathbf{r})]t} m(\mathbf{r}, 0). \tag{21}$$

For further treatment it is convenient to introduce the Laplace transform of the magnetization decay

$$\hat{M}(s) = \int_0^\infty \mathrm{d}t \mathrm{e}^{-st} M(t). \tag{22}$$

The relation between the Laplace transform $\hat{M}_0(s)$ of the static dephasing regime and the whole dynamic range is given by [3,5]

$$\begin{aligned} \hat{M}(s) &= \frac{1}{V} \int_V \mathrm{d}^3\mathbf{r} \frac{1}{s - \tau^{-1}(\hat{\Pi} - \hat{1}) - i\omega(\mathbf{r})} \\ &= \hat{M}_0(s + \tau^{-1}) + \tau^{-1} \cdot \hat{M}_0(s + \tau^{-1}) \cdot \hat{M}(s) \\ &= \frac{\hat{M}_0(s + \tau^{-1})}{1 - \tau^{-1} \cdot \hat{M}_0(s + \tau^{-1})}. \end{aligned} \tag{23}$$

In principle, the exact time course of the magnetization can be obtained by a Laplace backward transformation of Eq. (23). However in most cases this is not possible analytically. As shown in [9] a monoexponential signal decay is justified in the long time limit. For short echo times however a Gaussian shaped signal decay of the form $\sim\exp[-t^2\delta\omega^2\langle f(\mathbf{r})^2\rangle/2]$ occurs [8,9]. As in clinical applications short echo times are hard to realize and the transverse relaxation time is usually used to characterize tissue we focus on exponential decay. Beside this the non-Gaussian character of spin dephasing is well described by the strong collision approximation as shown in [9]. The strong collision approximation itself leads to a signal decay which includes the long and short time limit (see Eq. (38) in [9]). In many applications one is interested in the relaxation time T_2^* which approximates the situation best as a single exponential decay. This is given by the mean relaxation time T_2^* which is defined as [13]

$$T_2^* = \int_0^\infty \mathrm{d}t M(t) = \hat{M}(0). \tag{24}$$

Inserting Eq. (16) into Eq. (23) and applying Eq. (24), the dependence of T_2^* on τ as well as on $\delta\omega$ is found to be

$$\frac{1}{T_2^*(\tau, \delta\omega)} = \frac{1}{\frac{1}{V}\int_V \mathrm{d}^3\mathbf{r} \frac{1}{\tau^{-1} - i\delta\omega f(\mathbf{r})}} - \frac{1}{\tau}. \tag{25}$$

To obtain the scaling behavior we replace $\tau \to \lambda\tau$ and $\delta\omega \to \mu\delta\omega$ and find

$$\begin{aligned} \frac{1}{T_2^*(\lambda\tau, \mu\delta\omega)} &= \frac{1}{\frac{1}{V}\int_V \mathrm{d}^3\mathbf{r} \frac{1}{(\lambda\tau)^{-1} - i\mu\delta\omega f(\mathbf{r})}} - \frac{1}{\lambda\tau} \\ &= \frac{1}{\frac{\lambda}{\mu}\frac{1}{V}\int_V \mathrm{d}^3\mathbf{r} \frac{1}{(\mu\tau)^{-1} - i\lambda\delta\omega f(\mathbf{r})}} - \frac{\mu}{\lambda\mu\tau}. \end{aligned} \tag{26}$$

This directly implies the scaling law

$$T_2^*(\lambda\tau, \mu\delta\omega) = \frac{\lambda}{\mu} T_2^*(\mu\tau, \lambda\delta\omega), \tag{27}$$

where the scaling parameters λ and μ are arbitrary real numbers. Since we did not specify the geometry function this relation is valid for objects of the same shape but arbitrary parameters like size, susceptibility, external magnetic fields or diffusion coefficients of the surrounding medium.

These parameters determine the two scaled variables τ and $\delta\omega$ for a given shape. With Eq. (25) at hand we are able to give a mathematical criterion on the validity range of the strong collision approximation. As it is valid in the case that $\tau < T_2^*$. Using Eq. (8) in [16] and with the help of Eq. (25) we obtain the criterion

$$\mathrm{Re}\left[\left(\frac{1}{V}\int_V \mathrm{d}^3\mathbf{r}\,\frac{1}{1+i\tau\delta\omega f(\mathbf{r})}\right)^{-1}\right] < 2. \tag{28}$$

In the motional narrowing regime (where the relaxation time is given by $T_2^* \propto 1/\tau\delta\omega^2$) the condition of validity of the strong collision approximation $\tau < T_2^*$ leads to the condition $\tau^2\delta\omega^2 < 1$. In the opposite limit the static dephasing regime (where the relaxation time is given by $T_2^* \propto 1/\delta\omega$) the same condition results in $\tau\delta\omega < 1$. Thus it is possible to expand Eq. (28) up to the quadratic term in $\tau\delta\omega$ and we eventually arrive at

$$(\Delta f(\mathbf{r}))^2 = \langle f(\mathbf{r})^2\rangle - \langle f(\mathbf{r})\rangle^2 < \frac{1}{2\tau^2\delta\omega^2}, \tag{29}$$

where the expectation value of a function is defined in Eq. (9). The derived relation (29) can be used as a mathematical criterion to describe the validity of the strong collision approximation for a given shape of the magnetic perturber. Assuming an exponential decay of the spin echo signal we are able to use Eq. (24) in [4]. To obtain similar scaling laws for the spin echo relaxation time T_2 we use the relation

$$T_2 = T_2^* + \tau, \tag{30}$$

valid in the strong collision approximation, [4,5] which includes static dephasing as well as motional narrowing as limiting cases. In analogy to Eq. (27) we obtain the scaling law for the spin echo relaxation time T_2

$$T_2(\lambda\tau, \mu\delta\omega) = \frac{\lambda}{\mu}T_2(\mu\tau, \lambda\delta\omega). \tag{31}$$

It is important to note that Eqs. (25) and (30) are derived in the strong collision approximation. Thus, also the scaling laws are rigorous in this approximation only. Following the ideas of Weisskoff et al. [1] it is possible to derive from the Bloch-Torrey- Eq. (13) the relation

$$T_2(R, D, \delta\omega) = T_2(\lambda R, \lambda^2 D, \delta\omega). \tag{32}$$

This scaling law depends on a characteristic size R, the diffusion coefficient D and the characteristic frequency $\delta\omega$ which implies with Eq. (31) that the correlation time is a function of R and D. In the scaling law of Weisskoff $\delta\omega$ is kept constant while R and D are varied. This equals setting $\lambda = 1 = \mu$ in our scaling law Eq. (31) and implies that also τ must stay unaffected. Thus, we conclude that τ is a function of the ratio of R^2 and D

$$\tau = \tau(R^2/D). \tag{33}$$

To specify this dependence it is possible to apply the coordinate transform $\mathbf{r} = R\mathbf{u}$ (which corresponds to a dilation) to the Eq. (12) while keeping the volume fraction η unaffected. Taking into account that the form of the geometry function is independent of the variable when using this transform, i. e. $f(\mathbf{r}) = f(\mathbf{u})$ we find for the correlation time $\tau = k(\eta)R^2/D$. An analogous relation was postulated by Stables et al. (Eq. (15) in [17]). For a given geometry the dependence of k on η is given by

$$k(\eta) = \frac{1}{\langle f^2(\mathbf{u})\rangle V}\int_V \mathrm{d}^3\mathbf{u} f(\mathbf{u})\left(-\frac{1}{\nabla^2}\right)f(\mathbf{u}), \tag{34}$$

where all quantities and operators are according to the new variable $\mathbf{u}$. The implicit dependence on η is given by the regions of integration G and V in Eqs. (4) and (12).

4. Application to a capillary

In this section we will apply our results to a capillary modeled by a cylindrical magnetic inhomogeneity crossing the voxel. A single capillary is often used to describe signal formation in myocardium [3]. In cardiac diagnosis the transverse relaxation times can be used to distinguish between unaffected and affected myocardium. Recently in our group we exploited the dependence of the transverse relaxation times on parameters of the capillary system to characterize capillary recruitment in the myocardium of patients with stable angina due to single-vessel coronary artery disease [18]. Therefore measurements were performed using a combination of contrast agent and vasodilatator. The concept of this method is explained in Fig. 6 of [18]. Both, the concentration of the contrast agent and the concentration of the vasodilatator influence the transverse relaxation times, where the contrast agent affects the parameter $\delta\omega$ and the vasodilatator affects the radius R_C of the capillary. In the case of a high concentration of vasodilatator and contrast agent the underlying diffusion regime is the static dephasing regime while in the opposite case the motional narrowing regime is present. While the static dephasing regime has been studied extensively [19], the derived scaling laws are used to obtain more general results which will be applied in the prediction of experiments using combinations of contrast agents and vasodilatators. For this purpose a capillary with a tilt angle θ in an external magnetic field B_0 is considered. The susceptibility inside the capillary is given by χ_i corresponding to the concentration of the contrast agent while χ_e is the susceptibility of the surrounding medium (Fig. 2). The radius of the capillary R_C can be directly influenced by the concentration of the vasodilator. The inhomogeneous magnetic field induced by the susceptibility differences inside the voxel [20] follows from Eq. (1):

$$B(\mathbf{r}) = \frac{\chi_i - \chi_e}{2}B_0 \sin^2\theta R_C^2\frac{\cos 2\phi}{r^2}, \tag{35}$$

where the according parameters are defined in Fig. 2. As already stated the local resonance frequency $\omega(\mathbf{r})$ can be written in the form of Eq. (2), with the susceptibility term

$$\delta\omega = \gamma\frac{\chi_i - \chi_e}{2}B_0 \sin^2\theta \tag{36}$$

and the geometry function

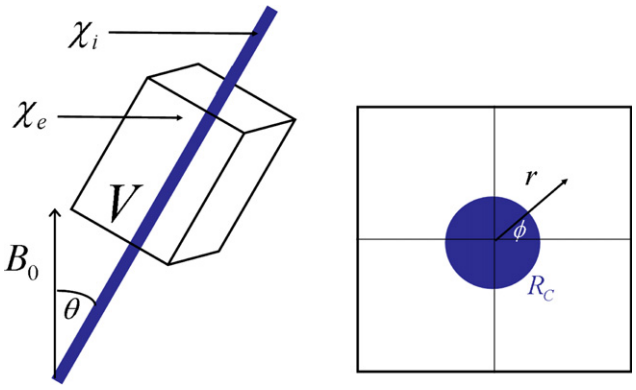

Fig. 2. (Left) Capillary in an external magnetic field. (Right) Cross sectional view of a voxel in polar coordinates.

$$f(\mathbf{r}) = R_C^2 \frac{\cos 2\phi}{r^2}. \tag{37}$$

In agreement with Eq. (33) the correlation time of the spins diffusing around the capillary is given by [3]

$$\tau = \frac{R_C^2}{4D} \frac{\ln \eta}{\eta - 1}, \tag{38}$$

where η is the volume fraction of the capillary inside the voxel. The relationship between the relaxation times and geometrical properties of the capillary had been investigated in previous publications [3,4,21,22] and results in

$$\frac{1}{\tau R_2} = 1 + \frac{1+\eta}{\left[\sqrt{1+(\eta\tau\delta\omega)^2}-1\right] + \eta\left[\sqrt{1+(\tau\delta\omega)^2}-1\right]}. \tag{39}$$

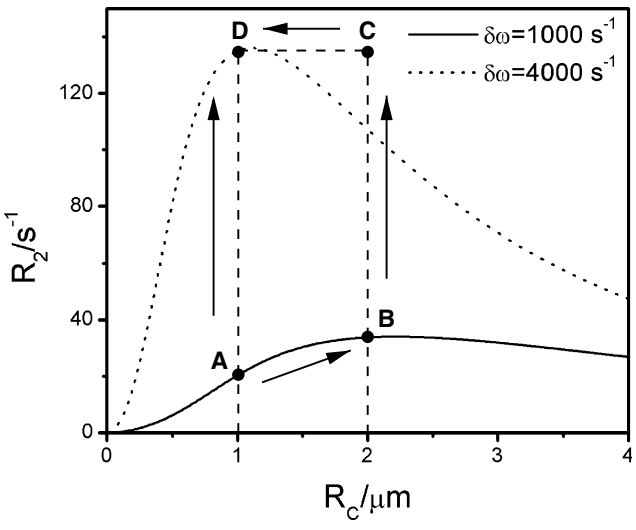

Fig. 3. Illustration of the scaling law $\lambda R_2(\sqrt{\lambda}R_C, \delta\omega) = R_2(R_C, \lambda\delta\omega)$ for the parameter $\lambda = 4$. The solid and the dashed curve are obtained from Eq. (39) for $D = 1\ \mu m^2\ ms^{-1}$ and $\eta = 0.05$. A doubling of the capillary radius (from point A to B) and following multiplication of the according relaxation rate with the scaling parameter $\lambda = 4$ (from point B to C) leads to the same value for the relaxation rate as a multiplication of the characteristic frequency $\delta\omega$ with the scaling parameter $\lambda = 4$ by keeping the primary radius unaffected (from point A to D).

With Eqs. (38) and (39) it is possible to relate the susceptibility shift $\Delta\chi$, the radius of the capillary R_C and the relaxation times T_2 and T_2^* with each other. To illustrate the scaling laws, the influence of an increase of the capillary radius R_C on the relaxation rate $R_2 = 1/T_2$ for various susceptibility shifts will be studied. With the general relationship Eq. (31) and the expression for the correlation time Eq. (38), the scaling law of the transverse relaxation rate yields $\lambda R_2(\sqrt{\lambda}R_C, \delta\omega) = R_2(R_C, \lambda\delta\omega)$ in the case of $\mu = 1$. This special case of the general scaling behavior coincides with the result of Weisskoff et al. (Eq. (9) in [1]). An application of this scaling law is visualized in Fig. 3.

5. Discussion

Starting from simple assumptions about the distribution of magnetic material inside a voxel, it is possible to derive simple scaling laws for the gradient echo relaxation time T_2^* as well as the spin echo relaxation time T_2 in the strong collision approximation. Thus it was possible to generalize previously obtained results. With these parameters one is able to predict effects of scaling the susceptibility shift, the external magnetic field, size of the perturber, or diffusion properties of the surrounding medium on the relaxation times. As already stated we see that the scaling behavior of the relaxation times as given in Eqs. (27) and (31) is valid for an arbitrary shape of the perturber. In contradiction to previously obtained results, it is possible to apply these scaling laws to arbitrary geometries, i.e. independently of the shape of the perturber it is possible to predict the scaling behavior of the transverse relaxation times. This enables us to connect different parameters of the problem in a quite simple way without considering the whole complexity of the system. In the special case of spheres this can be easily derived by directly inserting the scaling parameters in Eq. (8) in Ref. [16].

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